JOURNAL OF APPROXIMATION THEORY 12, 32–51 (1974)

The Numerical Evaluation by Splines of Fourier Transforms

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Communicated by Richard S. Varga

DEDICATED TO PROFESSOR I. J. SCHOENBERG
ON THE OCCASION OF HIS SEVENTIETH BIRTHDAY

INTRODUCTION

In [8], I. J. Schoenberg generalizes the construction of best quadrature formulas in two ways. He discusses integrals with an arbitrary pre-assigned weight function opening up the possibility of constructing this kind of q.f. for the numerical evaluation of Laplace transforms, Fourier integrals, and other special integral transforms. We pursue this possibility here; in particular, we wish to discuss approximations to the integrals

$$\int_{-\infty}^{\infty} f(x)\, e^{ixt}\, dx \tag{1}$$

$$\int_{0}^{\infty} f(x) \cos xt\, dx \tag{2}$$

$$\int_{0}^{\infty} f(x) \sin xt\, dx \tag{3}$$

In the paper [8], for m, a positive integer and $w(x)$, an arbitrary pre-assigned weight function, Schoenberg discusses q.f. of the form

$$\int_0^n w(x) f(x)\, dx = \sum_{\nu=0}^{n} H_{\nu,n}^{(m)} f(\nu) + \sum_{j=1}^{m-1} B_{j,n}^{(m)} f^{(j)}(0) + \sum_{j=1}^{m-1} C_{j,n}^{(m)} f^{(j)}(n) + Rf. \tag{4}$$

He requires: (i) that the q.f. (4) be exact, i.e., $Rf = 0$, if $f \in \pi_{m-1}$, the class of polynomials of degree not exceeding $m - 1$; and (ii) that the functional, Rf, when written in Peano-fashion as an integral of the form $\int_0^n K(x) f^{(m)}(x)\, dx$ has the kernel $K(x)$ with least L_2-norm. This q.f., he shows, is uniquely characterized by requiring $Rf = 0$ if f is a spline function of degree $2m - 1$

having the knots $1, 2, \ldots, n-1$, that is, $f(x) \in C^{2m-2}(\mathbb{R})$ and $f(x) \in \pi_{2m-1}$ for x in $(-\infty, 1), (1, 2), \ldots, (n-1, n), (n, \infty)$.

Here we shall discuss infinite analogs of the q.f. (4) for the real line $\mathbb{R}$ and the half-line $(0, \infty)$ or $\mathbb{R}^+$. We first consider the entire line, the so-called *cardinal* case when all the integers ν are nodes of the q.f. Let S_n (n a positive integer) denote the class of functions $S(x)$ such that

(i) $S(x) \in C^{n-2}(\mathbb{R})$;
(ii) $S(x) \in \pi_{n-1}$ in each interval $(\nu + n/2 - 1, \nu + n/2)$ for all integers ν.

Such functions are called *cardinal spline functions of degree* $n-1$.

Let n be even, say $n = 2m$, and consider a q.f. of the form

$$\int_{-\infty}^{\infty} f(x)\, dx = \sum_{-\infty}^{\infty} H_\nu^{(2m)} f(\nu) + Rf, \tag{5}$$

where the numerical coefficients $H_\nu^{(2m)}$ satisfy the condition that

$$| H_\nu^{(2m)} | < K \qquad \text{for all } \nu \text{ and some appropriate } K. \tag{6}$$

In [10, Theorem 5, p. 30] Schoenberg proves the following:

Among all quadrature formulas (5), (6), *the q.f.*

$$\int_{-\infty}^{\infty} f(x)\, dx = \sum_{-\infty}^{\infty} f(\nu) + Rf \tag{7}$$

is characterized by the requirement that $Rf = 0$ *iff* $f \in S_{2m} \cap L_1(\mathbb{R})$.

In Part I, we first consider the analog of the q.f. (4) for the entire line $\mathbb{R}$ and we take $w(x) = e^{ixt}$, that is, we discuss approximations to the Fourier transform (1). Let n be any positive integer and consider a q.f. of the form

$$\int_{-\infty}^{\infty} f(x)\, e^{ixt}\, dx = \sum_{-\infty}^{\infty} H_{\nu,t}^{(n)} f(\nu) + Rf \tag{8}$$

where the coefficients $H_{\nu,t}^{(n)}$ satisfy the condition that

$$| H_{\nu,t}^{(n)} | < K \text{ for fixed } t, \text{ for all } \nu, \text{ and some } K. \tag{9}$$

Note that the coefficients $H_{\nu,t}^{(n)}$ are now functions of t.

To describe our analog of (7) we need some notation introduced in [6, pp. 79, 114–116] and discussed further in Section 1, below. We define, for k a natural number,

$$\psi_k(t) = \left(\frac{2 \sin t/2}{t}\right)^k \tag{10}$$

and

$$\phi_k(t) = \sum_{j=-\infty}^{\infty} \psi_k(t + 2\pi j). \tag{11}$$

The $\phi_k(t)$ term is a cosine polynomial, positive for all real t [7, Lemma 6, p. 180]. We may now state

THEOREM 1. *Suppose $f(x) \in C^n(\mathbb{R})$, and that $f(x)$ and $f^{(n)}(x)$ are in $L_1(\mathbb{R})$ and $\to 0$ as $x \to \pm\infty$. Among all quadrature formulas of the form* (8), (9), *there is a unique formula, given by*

$$\int_{-\infty}^{\infty} f(x)\, e^{ixt}\, dx = \frac{\psi_n(t)}{\phi_n(t)} \sum_{-\infty}^{\infty} f(\nu)\, e^{i\nu t} + Rf \tag{12}$$

with the property

$$Rf = 0 \qquad \textit{whenever} \qquad f \in S_n \cap L_1(\mathbb{R}). \tag{13}$$

We obtain this q.f. (12) by using Newton's fundamental idea: assuming the function $f(x)$ to be given numerically at equidistant points of step 1, including the origin 0, we interpolate $f(x)$ by a function $S(x)$ at these points, and then construct the Fourier transform of $S(x)$. This idea has been used before, and often, for the integrals (1)–(3) [4]. In fact, for $n = 2$, the case of linear spline interpolation, the q.f. (12) can be found in [4, pp. 22, 23].

In Part I, we also consider the analog of the q.f. (4) for the half-line $\mathbb{R}^+$ and we take $w(x) = \cos xt$ or $w(x) = \sin xt$. With this choice of $w(x)$ and m a positive integer, we seek a q.f. of the form

$$\int_0^{\infty} w(x) f(x)\, dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(2m)} f(\nu) + \sum_{j=1}^{m-1} B_{j,t}^{(2m)} f^{(j)}(0) + Rf, \tag{14}$$

where the coefficients $H_{\nu,t}^{(2m)}$ satisfy the condition that

$$| H_{\nu,t}^{(2m)} | < K \text{ for fixed } t, \text{ all integers } \nu \geqslant 0, \text{ and some } K. \tag{15}$$

Again, for m fixed, the $H_{\nu,t}^{(2m)}$ are functions of t. Theorem 5 in Section 4 below gives an explicit form for these q.f.

We also consider q.f. of the form

$$\int_0^{\infty} f(x) \cos xt\, dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(2m)} f(\nu) + \sum_{j=1}^{m-1} B_{j,t}^{(2m)} f^{(2j-1)}(0) + Rf, \tag{16}$$

$$\int_0^{\infty} f(x) \sin xt\, dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(2m)} f(\nu) + \sum_{j=1}^{m-1} B_{j,t}^{(2m)} f^{(2j)}(0) + Rf, \tag{17}$$

where the coefficients again satisfy the condition (15). For the weight function $\cos xt$, we obtain the following:

THEOREM 2. *Suppose $f(x) \in C^{2m}(\mathbb{R}^+)$, and that $f(x)$ and $f^{(2m)}(x)$ are in $L_1(\mathbb{R}^+)$ and $\to 0$ as $x \to \infty$. Then, among all q.f. of the form* (16), (15) *there is a unique q.f., given by*

$$\int_0^\infty f(x) \cos xt \, dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \left\{ \frac{1}{2} f(0) + \sum_{\nu=1}^{\infty} f(\nu) \cos \nu t \right\}$$

$$+ \sum_{j=1}^{m-1} \frac{(-1)^j}{t^{2j}} \left[1 - \frac{\psi_{2m-2j}(t)\, \phi_{2j}(t)}{\phi_{2m}(t)} \right] f^{(2j-1)}(0) + Rf, \tag{18}$$

with the property

$$Rf = 0 \qquad \textit{whenever} \qquad f \in S_{2m} \cap L_1(\mathbb{R}^+). \tag{19}$$

The analogous theorem for the weight function $\sin xt$ is stated in Theorem 6, Section 4 below.

We obtain the q.f. (16) and (17) by constructing the cosine or sine transform of the appropriate spline interpolant. Closest to this point of view is the paper [1] in which Einarsson approximates integrals of the form

$$\int_a^b f(x) \cos wx \, dx, \qquad \int_a^b f(x) \sin wx \, dx$$

by taking the transform of a cubic spline with equidistant knots that matches $f(x)$ at the knots and the values $f'(a)$ and $f'(b)$ at the appropriate endpoints.

For the case of (16) and (17) for a finite interval, Marsden and Taylor in [5] exhibit precisely the analogues of the q.f. in Theorems 2 and 6. In fact, their results for the finite interval allow us to establish Theorems 2 and 6 for general m.

Part II contains expressions for the error, as well as estimates of bounds of these errors, for the approximations we make in the first part. We acquire these expressions by showing that we could have constructed our q.f. another way, by utilizing a particular monospline. In Section 6 below, we establish

THEOREM 3. *Suppose $f(x) \in C^{2m}(\mathbb{R}^+)$ and $f^{(2m)}(x)$, $f(x)$ are in $L_1(\mathbb{R}^+)$ and $\to 0$ as $x \to \infty$.*

1°. *The remainder Rf in the q.f.* (18) *of Theorem* 2 *is given by*

$$Rf = \frac{(-1)^m}{t^{2m}} \int_0^\infty [\cos xt - S_t(x)] f^{(2m)}(x) \, dx \tag{20}$$

where $S_t(x)$ is the unique, bounded $(2m - 1)$st *degree cardinal spline interpolating* $\cos xt$ *at the integers.*

2°. *For the step length h, we can bound Rf in the q.f.*

$$\int_0^\infty f(x)\cos xt\,dx = \frac{\psi_{2m}(th)}{\phi_{2m}(th)}\,h\left\{\frac{1}{2}f(0) + \sum_{\nu=1}^{\infty} f(\nu h)\cos \nu th\right\}$$
$$+ \sum_{j=1}^{m-1}\frac{(-1)^j}{t^{2j}}\left[1 - \frac{\phi_{2j}(th)\,\psi_{2m-2j}(th)}{\phi_{2m}(th)}\right] f^{(2j-1)}(0) + Rf \tag{21}$$

by

$$|Rf| \leqslant A_m\left(\frac{h}{\pi}\right)^{2m}\|f^{(2m)}\|_{L_1(\mathbb{R}^+)} \qquad \text{for} \quad -\pi/h \leqslant t \leqslant \pi/h \tag{22}$$

where

$$A_m = 2\left(1 + 2\sum_{\nu=1}^{\infty}\frac{1}{(2\nu+1)^{2m}}\right) < 3 \qquad \text{for} \quad m = 1, 2, \ldots \tag{23}$$

We use I. J. Schoenberg's very nice application of the exponential Euler splines to get our bound for the Fourier transform case, Theorem 7, Section 5 below, and show that this approach also gives us the bound in Theorem 3. The result is also an improvement over our original estimate in which the A_m of (22) was replaced by the number 4 [13, p. 91].

I. Approximations to the Transforms (1), (2), (3)

1. *Preliminaries.* We first recall some known definitions and results [6]. Let n be a natural number and define the central B-spline or basis spline

$$M(x) = M_n(x) = \frac{1}{(n-1)!}\,\delta^n x_+^{n-1}, \tag{1.1}$$

where

$$x_+ = \begin{cases} x & \text{if } x \geqslant 0, \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

where δ^n stands for the usual symbol for the nth order central difference of step equal to 1. $M_n(x)$ is a spline function of degree $n - 1$ having as knots the points ν (ν integer), or $\nu + \frac{1}{2}$, depending on whether $n - 1$ is odd or even. $M_n(x)$ is positive in the interval $(-\frac{1}{2}n, \frac{1}{2}n)$ and vanishes elsewhere, and evidently $M_n(x) \in S_n$. It has the following Fourier transform:

$$\int_{-\infty}^{\infty} M_n(x)\,e^{ixt} = \psi_n(t) \tag{1.2}$$

where

$$\psi_n(t) = \left(\frac{2 \sin \frac{t}{2}}{t} \right)^n. \tag{1.3}$$

(See [6], pp. 67–72).

We also define a *forward B-spline* $Q_n(x)$ by

$$Q_n(x) = M_n\left(x - \frac{n}{2}\right) = \frac{1}{(n-1)!} \sum_{i=1}^{n} (-1)^i \binom{n}{i} (x-i)_+^{n-1}. \tag{1.4}$$

$Q_n(x)$ has integer knots, is positive in $(0, n)$ and zero elsewhere.

With $\psi_n(t)$ defined by (1.3), we define

$$\phi_n(t) = \sum_{j=-\infty}^{\infty} \psi_n(t + 2\pi j). \tag{1.5}$$

$\phi_n(t)$ is a cosine polynomial of period 2π and order $[(n+1)/2] - 1$ that can be explicitly computed from the expression

$$\phi_n(t) = \sum_{\nu=-\infty}^{\infty} M_n(\nu)\, e^{i\nu t} = \sum_{|\nu| \leqslant n/2} M_n(\nu) \cos \nu t. \tag{1.6}$$

Related to the equivalent form

$$\phi_n(t) = \left(2 \sin \frac{t}{2}\right)^n \sum_{\nu} \frac{(-1)^{\nu n}}{(t + 2\pi\nu)^n},$$

we define a new set of periodic functions by

$$\rho_n(t) = \left(2 \sin \frac{t}{2}\right)^n \sum_{\nu} \frac{1}{(t + 2\pi\nu)^n}.$$

Evidently

$$\phi_n(t) = \rho_n(t) \text{ if } n \text{ is even.} \tag{1.7}$$

The functions (1.7) can be obtained recursively from

$$\rho_{n+1}(t) = \cos \frac{t}{2}\, \rho_n(t) - \frac{2}{n} \sin \frac{t}{2}\, \rho_n{}'(t) \tag{1.8}$$

starting with the initial value $\rho_1(t) = \cos(t/2)$ (see [6], pp. 114, 115).

By Lemma 6 of [7, p. 180] we have

$$\max_t \phi_n(t) = \phi_n(0) = 1, \qquad \min_t \phi_n(t) = \phi_n(\pi) > 0. \tag{1.9}$$

By (1.6) we find

$$\phi_2(t) = 1,$$
$$\phi_3(t) = (3 + \cos t)/4,$$
$$\phi_4(t) = (2 + \cos t)/3.$$

We shall need $\mathscr{L}_n(x)$, the so-called fundamental cardinal spline function of order n, or degree $n - 1$, which we call the unique, bounded member of S_n which satisfies

$$\mathscr{L}_n(\nu) = \delta_{0\nu} \qquad \text{for all integers } \nu.$$

It was found [6, p. 124] that

$$\mathscr{L}_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi_n(t)}{\phi_n(t)} e^{-ixt}\, dx$$

and, inverting, that

$$\int_{-\infty}^{\infty} \mathscr{L}_n(x)\, e^{ixt}\, dx = \frac{\psi_n(t)}{\phi_n(t)}. \tag{1.10}$$

2. *Proof of Theorem* 1. The author is indebted to I. J. Schoenberg for the following simplified version of the proof. We first note that the result [10, p. 27; 13, p. 12]

$$S(x) \in S_n \cap L_1(\mathbb{R}) \text{ implies } \sum_{-\infty}^{\infty} |S(\nu)| < \infty$$

and (9) guarantee that the functional Rf is well defined by (8) if $S(x) \in S_n \cap L_1(\mathbb{R})$.

To derive the q.f. (12), we start from the identity

$$S(x) = \sum_{\nu=-\infty}^{\infty} S(\nu)\, \mathscr{L}_n(x - \nu) \tag{2.1}$$

valid for any cardinal spline of power growth, $S(x) \in S_n$ [9, Theorem 3, p. 407]. In particular, this identity is valid if $S(x) \in S_n \cap L_1(\mathbb{R})$. Multiplying by e^{ixt} and integrating, we find

$$\int_{-\infty}^{\infty} S(x)\, e^{ixt}\, dx = \sum_{\nu=-\infty}^{\infty} A_{\nu,t}^{(n)} S(\nu), \tag{2.2}$$

where

$$A_{\nu,t}^{(n)} = \int_{-\infty}^{\infty} \mathscr{L}_n(x - \nu)\, e^{ixt}\, dx = e^{i\nu t} \int_{-\infty}^{\infty} \mathscr{L}_n(x)\, e^{ixt}\, dx$$

or, by (1.10),

$$A_{\nu,t}^{(n)} = \frac{\psi_n(t)}{\phi_n(t)} e^{i\nu t}.$$

The interchange of the integral and the sum in (2.2) is justified because $S(x) \in S_n \cap L_1(\mathbb{R})$ and the series in (2.1) converges absolutely and locally uniformly. By (1.9), we obtain

$$\left| \frac{\psi_n(t)}{\phi_n(t)} e^{i\nu t} \right| \leqslant \frac{1}{\phi_n(0)},$$

so that (9) is satisfied.

In order to characterize the q.f. (12), we suppose (8), (9) and (13) hold. If we choose $f(x) = \mathscr{L}_n(x - \nu)$, we obtain

$$Rf = 0 = \int_{-\infty}^{\infty} \mathscr{L}_n(x-\nu)\, e^{ixt}\, dx - H_{\nu,t}^{(n)},$$

or

$$H_{\nu,t}^{(n)} = \int_{-\infty}^{\infty} \mathscr{L}_n(x-\nu)\, e^{ixt}\, dx = \frac{\psi_n(t)}{\phi_n(t)} e^{i\nu t}$$

as above.

3. *Proof of Theorem* 2. We take $w(x) = \cos xt$, but consider more general boundary conditions that include those of Theorem 2 as a special case. Partition the numbers $1, 2,\ldots, 2m - 2$ into the $m - 1$ disjoint pairs.

$$(1, 2m-2), (2, 2m-3), \ldots, (m-1, m). \tag{3.1}$$

Note that the sum of the numbers in each pair is $2m - 1$. Let I be a set of $m - 1$ numbers obtained by choosing one and only one number from each of the pairs (3.1). One possible choice is

$$I = \{1, 3, 5, \ldots, 2m-3\} \tag{3.2}$$

which corresponds to the derivative data required in Theorem 2.

For simplicity, we write $H_\nu = H_{\nu,t}^{(2m)}$, $B_i = B_{i,t}^{(2m)}$. We want to construct a q.f. of the form

$$\int_0^\infty f(x) \cos xt\, dx = \sum_{\nu=0}^{\infty} H_\nu f(\nu) + \sum_{i \in I} B_i f^{(i)}(0) + Rf \tag{3.3}$$

such that

$$|H_\nu| < K \text{ for all integers } \nu \geqslant 0 \text{ and some } K \tag{3.4}$$

and with the property that

$$Rf = 0 \qquad \text{if} \quad f \in S_{2m} \cap L_1(\mathbb{R}^+). \tag{3.5}$$

We shall need the fundamental functions

$$L_\nu(x) \quad (\nu = 0, 1,...); \qquad \Lambda_i(x) \quad (i \in I)$$

for this semicardinal case. There are the unique, bounded members of $S_{2m} \cap L_1(\mathbb{R}^+)$ that satisfy

$$L_\nu(j) = \delta_{\nu j}, \qquad L_\nu^{(i)}(0) = 0 \qquad (i \in I) \tag{3.6}$$

$$\Lambda_i(\nu) = 0, \qquad \Lambda_i^{(k)}(0) = \delta_{ik} \qquad (i, k \in I) \tag{3.7}$$

Schoenberg [11, Theorem 2, p. 86] has shown that

$$|L_\nu(x)| < Ae^{-\alpha|x-\nu|}, \qquad |\Lambda_i(x)| < Ae^{-\alpha x} \quad \text{for } x \geqslant 0 \tag{3.8}$$

for appropriate constants A and α depending only on m.

The proof for the semicardinal case proceeds the same way as the proof of the cardinal case, Theorem 1. The result [10, p. 27]

$$S(x) \in S_{2m} \cap L_1(\mathbb{R}^+) \text{ implies } \sum_0^\infty |S(\nu)| < \infty$$

and (3.5) guarantee that Rf is well defined by (3.3) if $S(x) \in S_{2m} \cap L_1(\mathbb{R}^+)$. This time we begin with the identity

$$S(x) = \sum_{\nu=0}^{\infty} S(\nu)\, L_\nu(x) + \sum_{i \in I} S^{(i)}(0)\, \Lambda_i(x) \tag{3.9}$$

valid for any semicardinal spline that satisfies $S(x) = O(x^\gamma)$ as $x \to \infty$, [11, Theorem 3, p. 86]. Taking the cosine transform of $S(x)$ gives

$$\int_0^\infty S(x) \cos xt\, dx = \sum_{\nu=0}^{\infty} A_\nu S(\nu) + \sum_{i \in I} C_i f^{(i)}(0) \tag{3.10}$$

where

$$A_\nu = \int_0^\infty L_\nu(x) \cos xt\, dx, \qquad C_i = \int_0^\infty \Lambda_i(x) \cos xt\, dx \tag{3.11}$$

The interchange of integral and sum in (3.10) is justified as in the proof of Theorem 1.

To establish the unicity of the q.f. we suppose that (3.3), (3.4), and (3.5) hold and choose $f(x)$ equal to $L_\nu(x)$ and then $\Lambda_i(x)$ to obtain

$$\begin{aligned} H_\nu &= A_\nu = \int_0^\infty L_\nu(x) \cos xt \, dx \\ B_i &= C_i = \int_0^\infty \Lambda_i(x) \cos xt \, dx, \end{aligned} \tag{3.12}$$

respectively, because of (3.6), (3.7), and (3.11). We have established the following:

THEOREM 4. *Suppose* $f(x) \in C^{2m}(\mathbb{R}^+)$ *and* $f^{(2m)}(x)$ *are in* $L_1(\mathbb{R}^+)$ *and* $\to 0$ *as* $x \to \infty$. *Among all q.f. of the form* (3.3), (3.4), *there is a unique q.f., given by* (3.3) *and* (3.12), *that satisfies* (3.5).

For the choice of $I = \{1, 3, 5,..., 2m - 3\}$ corresponding to the derivatives needed in Theorem 2, we can obtain simple, explicit forms for the coefficients H_ν, B_i. We employ some results that Marsden and Taylor [5] obtained for the finite interval. To do this, we restrict any $S(x) \in S_{2m} \cap L_1(\mathbb{R}^+)$ to the interval $[0, N]$, obtaining a spline function of degree $2m - 1$ on $[0, N]$, for which [5, pp. 1, 8, 11]

$$\begin{aligned} \int_0^N S(x) \cos xt \, dx = {} & \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \left\{ \frac{1}{2} S(0) + \sum_{k=1}^{N-1} S(k) \cos kt + \frac{1}{2} S(N) \cos Nt \right\} \\ & + \sum_{j=1}^{m-1} \frac{(-1)^j}{t^{2j}} \left[1 - \frac{\psi_{2m-2j}(t)\, \rho_{2j}(t)}{\phi_{2m}(t)} \right] S^{(2j-1)}(0) \\ & - \sum_{j=1}^{m-1} \frac{(-1)^j}{t^{2j}} \left[1 - \frac{\psi_{2m-2j}(t)\, \rho_{2j}(t)}{\phi_{2m}(t)} \right] S^{(2j-1)}(N) \cos Nt \\ & - \sum_{j=0}^{m-1} \frac{(-1)^{j+1}}{t^{2j+1}} \left[1 - \frac{\psi_{2m-2j-1}(t)\, \rho_{2j+1}(t)}{\phi_{2m}(t)} \right] S^{(2j)}(N) \sin Nt. \end{aligned} \tag{3.13}$$

Note that if we use Markov's Theorem repeatedly and (3.8), we obtain

$$L_\nu^{(k)}(N) \to 0, \qquad \Lambda_i^{(k)}(N) \to 0 \qquad \text{as} \quad N \to \infty \qquad (k = 0, 1,..., 2m - 2). \tag{3.14}$$

Now, let $S(x) = L_\nu(x)$ in (3.13), so that, by (3.6) and (3.14) we obtain

$$\int_0^\infty L_0(x) \cos xt\, dx = \frac{1}{2} \frac{\psi_{2m}(t)}{\phi_{2m}(t)},$$

$$\int_0^\infty L_\nu(x) \cos xt\, dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \cos \nu t \qquad (\nu = 1, 2,\ldots).$$

Similarly, if we substitute $\Lambda_i(x)$ for $S(x)$ in (3.13), we find

$$\int_0^\infty \Lambda_{2j-1}(x) \cos xt\, dx = \frac{(-1)^j}{t^{2j}} \left[1 - \frac{\psi_{2m-2j}(t)\, \rho_{2j}(t)}{\phi_{2m}(t)}\right] \quad (j = 1, 2,\ldots, m-1)$$

These results, together with Theorem 4, prove Theorem 2.

4. *Different Derivative Data*; *the Sine Transform.* We can also express the q.f. of Theorem 4 in another way that more explicitly relates this q.f. to the one of Theorem 2. For $I = \{1, 2,\ldots, m-1\}$ we have

THEOREM 5. *Let $S(x) \in L_1(\mathbb{R}^+)$ be the unique spline of degree $2m-1$ for $x \geqslant 0$ with knots $x = 1, 2,\ldots$ satisfying the conditions*

$$\begin{aligned} S(\nu) &= f(\nu) \qquad && (\nu = 0, 1, 2,\ldots) \\ S^{(i)}(0) &= f^{(i)}(0) \qquad && (i = 1, 2,\ldots, m-1). \end{aligned} \tag{4.1}$$

Then the q.f. of Theorem 3 may be written as

$$\begin{aligned} \int_0^\infty f(x) \cos xt\, dx = {} & \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \left\{\frac{1}{2} f(0) + \sum_{\nu=1}^{\infty} f(\nu) \cos \nu t\right\} \\ & + \sum_{2i-1 \leqslant m-1} \frac{(-1)^i}{t^{2i}} \left[1 - \frac{\psi_{2m-2i}(t)\, \phi_{2i}(t)}{\phi_{2m}(t)}\right] f^{(2i-1)}(0) \\ & + \sum_{2i-1 \geqslant m} \frac{(-1)^i}{t^{2i}} \left[1 - \frac{\psi_{2m-2i}(t)\, \phi_{2i}(t)}{\phi_{2m}(t)}\right] S^{(2i-1)}(0) + Rf. \end{aligned} \tag{4.2}$$

A proof follows from observing that the q.f. (18) of Theorem 2 is exact for the $S(x)$ of the hypothesis and from there applying (4.1).

In Section 3, we considered $w(x) = \cos xt$. Now we take $w(x) = \sin xt$ and indicate an analog of Theorem 2. Theorem 4 and its proof are valid if $\sin xt$ replaces $\cos xt$, and we get the coefficients of the q.f. in a particularly simple form if we now choose $I = \{2, 4,\ldots, 2m-2\}$. A similar expression to (3.13) [5, pp. 1, 8, 11] yields the following:

THEOREM 6. *Suppose $f(x) \in C^{2m}(\mathbb{R}^+)$ and $f(x)$ and $f^{(2m)}(x)$ are in $L_1(\mathbb{R}^+)$ and $\to 0$ as $x \to \infty$. Among all q.f. of the form*

$$\int_0^\infty f(x) \sin xt \, dx = \sum_{\nu=0}^{\infty} H_{\nu,t}^{(2m)} f(\nu) + \sum_{j=1}^{m-1} B_{j,t}^{(2m)} f^{(2j)}(0) + Rf,$$

where

$$| H_{\nu,t}^{(2m)} | < K \text{ for fixed } t, \text{ for all integers } \nu \geqslant 0 \text{ and some } K,$$

there is a unique q.f., given by

$$\int_0^\infty f(x) \sin xt \, dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \sum_{\nu=1}^{\infty} f(\nu) \sin \nu t$$
$$+ \sum_{j=0}^{m-1} \frac{(-1)^j}{t^{2j+1}} \left[1 - \frac{\psi_{2m-1-2j}(t)\, \rho_{2j+1}(t)}{\phi_{2m}(t)}\right] f^{(2j)}(0) + Rf$$

with the property

$$Rf = 0 \qquad \text{whenever} \qquad f \in S_{2m} \cap L_1(\mathbb{R}^+).$$

We also obtain an obvious analog of Theorem 5 for the sin xt case if we use Theorems 4 and 6.

II. EXPRESSIONS FOR THE ERRORS

5. *Exponential Euler Splines and the Fourier Transform.* In the introduction we mentioned that we could have constructed our q.f. in another way, by using a particular monospline. We do this here, and instead of basing our error bounds on a result of M. Golomb [3, p. 41] as in [13], we follow I. J. Schoenberg in developing our error expressions by using exponential Euler splines and their properties [see 11, Lecture 10, Part 1].

We first discuss the simpler, cardinal case, and consider the unique, bounded cardinal spline of degree $2m - 1$ interpolating e^{ixt} at the integers [11, Theorem 1, p. 85]. This spline

$$S(x) = S_{2m-1}(x; e^{it})$$

is also an exponential Euler spline where the base of the exponential is

$$y = e^{it} \neq 1, \quad \text{i.e.,} \quad S(x + 1) = yS(x) = e^{it}S(x).$$

We shall need more information about $S(x)$ in the interval $[0, 1]$. (See [11], pp. 21–24].) Suppose $0 \leqslant x \leqslant 1$ and $n = 2m - 1$ and define

$$\Phi_n(x; y) = \sum_j y^j Q_{n+1}(x - j) \tag{5.1}$$

where $Q_n(x)$ is the forward B-spline defined in (1.4), so that

$$S_n(x; y) = \Phi_n(x; y)/\Phi_n(0; y) \tag{5.2}$$

We also define the monic polynomial $A_n(x; y) = x^n +$ (lower degree terms) by

$$A_n(x; y) = n!\,(1 - y^{-1})^{-n}\, \Phi_n(x; y) \qquad (y \neq 0, y \neq 1). \tag{5.3}$$

This polynomial is also given by

$$A_n(x, y) = x^n + \binom{n}{1} a_1(y)\, x^{n-1} + \binom{n}{2} a_2(y)\, x^{n-2} + \cdots + a_n(y), \tag{5.4}$$

where

$$a_n(y) = (y - 1)^{-n}\, \Pi_n(y).$$

Here the $\Pi_n(y)$ are the so-called Euler-Frobenius polynomials. They are related to the forward B-spline $Q_{n+1}(x)$ by the identity

$$\Pi_n(y) = n! \sum_{j=0}^{n-1} Q_{n+1}(j + 1)\, y^j. \tag{5.5}$$

They also satisfy the recurrence relation

$$\Pi_{n+1}(y) = (1 + ny)\, \Pi_n(y) + y(1 - y)\, \Pi_n{}'(y) \qquad (\Pi_0(y) = 1) \tag{5.6}$$

from which we find

$$\begin{aligned} \Pi_0(y) &= 1, & \Pi_2(y) &= y + 1, \\ \Pi_1(y) &= 1, & \Pi_3(y) &= y^2 + 4y + 1. \end{aligned}$$

Later, we shall need information about derivatives of $S(x)$ so we record here that

$$A_n^{(j)}(0; y) = n!(y - 1)^{-n+j}\, \Pi_{n-j}(y)/(n - j)!$$

and (5.3) imply that

$$\Phi_n^{(j)}(0; y) = y^{-n}(y - 1)^j\, \Pi_{n-j}(y)/(n - j)! \qquad (j = 0, 1,\ldots, n) \tag{5.7}$$

We also find that

$$\Phi_n(0; y) = y^{-n}\Pi_n(y)/n! = y^{-n} \sum_{j=0}^{2m-2} Q_{2m}(j+1)\, y^j$$

$$= e^{-imt} \sum_{j=-(m-1)}^{m-1} M_{2m}(j)\, e^{ijt} = e^{-imt}\phi_{2m}(t) \tag{5.8}$$

Now suppose $f(x) \in C^{2m}(\mathbb{R})$ and that $f(x)$ and $f^{(2m)}(x)$ are in $L_1(\mathbb{R})$ and $\to 0$ as $x \to \pm\infty$. Let

$$K(x) = e^{ixt} - S_{2m-1}(x; e^{it}) \tag{5.9}$$

and consider the functional

$$Rf = \int_{-\infty}^{\infty} K(x) f^{(2m)}(x)\, dx = \int_{-\infty}^{\infty} K(x)\, df^{(2m-1)}(x).$$

Integrating this expression by parts iteratively yields

$$Rf = (-1)^{2m-1} \int_{-\infty}^{\infty} K^{(2m-1)}(x)\, df(x) = -\int_{-\infty}^{\infty} [(it)^{2m-1}e^{ixt} - S^{(2m-1)}(x)]\, df(x).$$

From this, we obtain

$$\int_{-\infty}^{\infty} f(x)\, e^{ixt}\, dx = (it)^{-2m} \sum_{-\infty}^{\infty} [S^{(2m-1)}(\nu+0) - S^{(2m-1)}(\nu-0)]\, f(\nu)$$

$$+ (it)^{-2m} \int_{-\infty}^{\infty} K(x) f^{(2m)}(x)\, dx \tag{5.10}$$

and require the following lemma proved by Schoenberg [11, Lecture 10].

LEMMA 1. *We have the relations*

$$(it)^{-2m}[S^{(2m-1)}(\nu+0) - S^{(2m-1)}(\nu-0) = \frac{\psi_{2m}(t)}{\phi_{2m}(t)}\, e^{i\nu t} \qquad \text{for all } \nu. \tag{5.11}$$

This shows that the coefficients of (5.10) are identical with those of (12), Theorem 1.

We now want to express the q.f. for the Fourier transform in Theorem 1 for $n = 2m$ in steps of length h. If we replace $f(x)$ in (12) or (5.10) by $f(xh)$ we obtain the relation

$$\int_{-\infty}^{\infty} f(xh)\, e^{ixt}\, dx = \frac{\psi_{2m}(t)}{\phi_{2m}(t)} \sum_{\nu} f(\nu h)\, e^{i\nu\tau} + Rf$$

where

$$Rf = (it)^{-2m} h^{2m} \int_{-\infty}^{\infty} [e^{ixt} - S(x; e^{it})] f^{(2m)}(xh)\, dx$$

Replacing first x by x/h and then replacing t by th yields

$$\int_{-\infty}^{\infty} f(x)\, e^{ixt}\, dx = \frac{\psi_{2m}(th)}{\phi_{2m}(th)}\, h \sum_{\nu} f(\nu)\, e^{i\nu th} + Rf \tag{5.12}$$

where

$$Rf = (it)^{-2m} \int_{-\infty}^{\infty} [e^{ixt} - S(x/h, e^{ith})] f^{(2m)}(x)\, dx. \tag{5.13}$$

Here $S(x/h, e^{ith})$ is the unique, bounded $(2m - 1)$st degree spline interpolating e^{ixt} at the points $x = 0, \pm h, \pm 2h, \ldots$.

We require Theorem 8 of Schoenberg's [11, p. 30] whose statement is the following:

If $-\pi \leqslant t \leqslant \pi$, then

$$| e^{ixt} - S_{2m-1}(x; e^{it})| \leqslant A_m \left(\frac{t}{\pi}\right)^{2m} \qquad \text{for all real } x, \tag{5.14}$$

where

$$A_m = 2\left(1 + 2 \sum_{\nu=1}^{\infty} \frac{1}{(2\nu + 1)^{2m}}\right) < 3 \qquad \text{for} \quad m = 1, 2, \ldots \tag{5.15}$$

This is applied by keeping the old t fixed and choosing $h > 0$ so that

$$-\pi/h \leqslant t \leqslant \pi/h. \tag{5.16}$$

Then, apply (5.14) with t replaced by th, and x by x/h, to obtain

$$| e^{ixt} - S_{2m-1}(x/h; e^{ith})| \leqslant A_m \left(\frac{th}{\pi}\right)^{2m} \qquad \text{for all } x. \tag{5.17}$$

Then (5.17) and (5.13) yield

$$| Rf | \leqslant A_m \left(\frac{h}{\pi}\right)^{2m} \| f^{(2m)} \|_{L_1(\mathbb{R})} \tag{5.18}$$

This establishes

THEOREM 7. *Suppose* $f^{(2m)}(x)$, $f(x)$ *are in* $L_1(\mathbb{R})$ *and* $\to 0$ *as* $x \to \pm\infty$. *Then we can bound* Rf *in the q.f.* (5.12) *by* (5.18) *for all* t *and* h *satisfying* (5.17).

6. *Proof of Theorem* 3. We can now adapt Schoenberg's proof of the cardinal case to the semicardinal case. We begin with the same monospline (5.9) we employed for the cardinal case, but now integrate the functional

$$Rf = \int_0^\infty K(x) f^{(2m)}(x)\, dx = \int_0^\infty K(x)\, df^{(2m-1)}(x)$$

by parts until eventually we find

$$\begin{aligned}\int_0^\infty f(x)\, e^{ixt}\, dx = &-(it)^{-2m} \sum_{j=1}^{2m-1} (-1)^j [(it)^{2m-1-j} - S^{(2m-1-j)}(0)] f^{(j)}(0)\\ &- (it)^{-2m}[(it)^{2m-1} - S^{(2m-1)}(0+0)] f(0)\\ &+ (it)^{-2m} \sum_{\nu=1}^{\infty} [S^{(2m-1)}(\nu+0) - S^{(2m-1)}(\nu-0)] f(\nu)\\ &+ (it)^{-2m} \int_0^\infty K(x) f^{(2m)}(x)\, dx. \end{aligned} \tag{6.1}$$

In addition to Lemma 1 we also need

LEMMA 2. *We have the relations*

$$\begin{aligned}1°.\quad &-(it)^{-2m}(-1)^j[(it)^{2m-1-j} - S^{(2m-1-j)}(0)]\\ &= (it^{-1})^{j+1} \left[1 - \frac{\psi_{2m-1-j}(t)\, \rho_{j+1}(t)}{\phi_{2m}(t)}\right] \end{aligned}\tag{6.2}$$

and

$$\begin{aligned}2°.\quad &-(it)^{-2m}[(it)^{2m-1} - S^{(2m-1)}(0+0)]\\ &= \frac{1}{2} \frac{\psi_{2m}(t)}{\phi_{2m}(t)} + (it^{-1}) \left[1 - \frac{\psi_{2m-1}(t)\, \rho_1(t)}{\phi_{2m}(t)}\right] \end{aligned}\tag{6.3}$$

Proof. For 1°, we first claim that, for $y = e^{it}$,

$$y^{-j}\Pi_j(y)/j! = e^{-[(j+1)/2]it} \rho_{j+1}(t) \qquad (j = 1, 2,...) \tag{6.4}$$

For $j = 1$, we get $y^{-1} = e^{-it}$. The rest follows by a straightforward induction using the recurrence relations (5.6), (1.8) that the $\Pi_j(y)$ and $\rho_{j+1}(t)$ satisfy. From (5.2), (5.7), and (6.4) we find

$$S^{(2m-1-j)}(0) = \Phi^{(2m-1-j)}(0, y)/\Phi(0, y) \tag{6.5}$$

where

$$\Phi^{(2m-1-j)}(0, y) = y^{-(2m-1)}(y-1)^{2m-1-j}\,\Pi_j(y)/j!$$
$$= (1-e^{-it})^{2m-1-j}\,e^{-[(j+1)/2]it}\rho_{j+1}(t) \tag{6.6}$$

The left side of (6.2) becomes

$$-(it)^{-(j+1)}(-1)^j[1-(it)^{-(2m-1-j)}\,S^{(2m-1-j)}(0)]$$
$$= (it^{-1})^{j+1}\left[1-\frac{\psi_{2m-1-j}(t)\,\rho_{j+1}(t)}{\phi_{2m}(t)}\right]$$

by successively using (6.5), (6.6) and (5.8).

For 2°, we note that because of (5.7) and (5.8) we have that

$$S^{(2m-1)}(0+0) = (1-e^{-it})^{2m-1}\,e^{imt}/\phi_{2m}(t)$$

so that the left side of (6.3) becomes

$$-(it)^{-1}[1-(it)^{-(2m-1)}(1-e^{-it})^{2m-1}\,e^{[(2m-1)/2]it}e^{-[(2m-1)/2]it}e^{imt}/\phi_{2m}(t)]$$
$$= (it^{-1})\left[1-(it)^{-(2m-1)}(1-e^{-it})^{2m-1}\,e^{[(2m-1)/2]it}\cos\frac{t}{2}\Big/\phi_{2m}(t)\right]$$
$$-(it^{-1})\left[(it)^{-(2m-1)}(1-e^{-it})^{2m-1}\,e^{[(2m-1)/2]it}\left(i\sin\frac{t}{2}\right)\Big/\phi_{2m}(t)\right],$$

which is the right side of (6.3).

If we use Lemmas 1 and 2 and take real and imaginary parts of (6.1), we obtain a q.f. for $\int_0^\infty f(x)\cos xt$ whose coefficients are identical with those of (18), Theorem 2 and a q.f. for $\int_0^\infty f(x)\sin xt\,dx$ whose coefficients are the same as those of (4.3), Theorem 6. We remark that $\operatorname{Re} S_{2m-1}(x; e^{it})$ and $\operatorname{Im} S_{2m-1}(x; e^{it})$ are the unique, bounded $(2m-1)$st degree cardinal splines interpolating $\cos xt$ and $\sin xt$, respectively, at the integers. So the first part of Theorem 3 is established.

The second part of Theorem 3 follows just as Theorem 7 does from the discussion in Section 5. In particular, the estimate (22) is a consequence of (5.17). The analogous result for the $\sin xt$ case is given by the following:

THEOREM 8. *Suppose $f(x) \in C^{2m}$ and that $f^{(2m)}(x)$ and $f(x)$ are in $L_1(\mathbb{R}^+)$ and $\to 0$ as $x \to \infty$.*

1°. *The remainder Rf in the q.f.* (4.3) *of Theorem* 6 *is given by*

$$Rf = \frac{(-1)^m}{t^{2m}}\int_0^\infty [\sin xt - \operatorname{Im} S_{2m-1}(x; e^{ixt})]\,f^{(2m)}(x)\,dx.$$

2°. *For the step length h, we can bound Rf in the q.f.*

$$\int_0^\infty f(x) \sin xt\, dx = \frac{\psi_{2m}(th)}{\phi_{2m}(th)} h \sum_{\nu=1}^{\infty} f(\nu h) \sin \nu th$$
$$+ \sum_{j=0}^{m-1} \frac{(-1)^j}{t^{2j+1}} \left[1 - \frac{\psi_{2m-1-2j}(th)\, \rho_{2j+1}(th)}{\phi_{2m}(th)}\right] f^{(2j)}(0) + Rf$$

by

$$|Rf| \leqslant A_m \left(\frac{h}{\pi}\right)^{2m} \|f^{(2m)}\|_{L_1(\mathbb{R}^+)} \qquad \text{for} \quad \frac{-\pi}{h} \leqslant t \leqslant \frac{\pi}{h}$$

where A_m *is given by* (5.15).

7. *Remarks.* In [2] Einarsson compares several methods for computing cosine transforms for the special case of $f(x) = e^{-x}$. One method he uses and the reason for the paper is based on the approximation of $f(x)$ by a cubic spline. This q.f., precisely the same one as (21) of Theorem 3 for $m = 2$, is

$$\int_0^\infty f(x) \cos xt\, dx = \frac{\psi_4(th)}{\phi_4(th)} h \left\{\frac{1}{2} f(0) + \sum_{\nu=1}^{\infty} f(\nu h) \cos \nu th\right\}$$
$$- \frac{1}{t^2} \left[1 - \frac{\psi_2(th)}{\phi_4(th)}\right] f'(0) + Rf \tag{7.1}$$

where we have used $\phi_2(t) = 1$. Einarsson's main conclusion is that this spline q.f. is superior to Filon's formula, a q.f. based on approximation of the function by a quadratic in each double interval and one of the most used formulas for the calculation of Fourier integrals.

Einarsson's calculations indicate that for small values of t, the q.f. (7.1) gives a relative error that is four times less than the Filon formula. For large values of t, the relative error of the Filon formula increases rapidly, while the spline method (7.1) gives a surprisingly small error growth. This same phenomenon we found to be the case for the following q.f., obtained from (21) for $m = 3$.

$$\int_0^\infty f(x) \cos xt\, dx = \frac{\psi_6(th)}{\phi_6(th)} h \left\{\frac{1}{2} f(0) + \sum_{\nu=1}^{\infty} f(\nu h) \cos \nu th\right\}$$
$$- \frac{1}{t^2} \left[1 - \frac{\psi_4(th)}{\phi_6(th)}\right] f'(0)$$
$$+ \frac{1}{t^4} \left[1 - \frac{\phi_4(th)\, \psi_2(th)}{\phi_6(th)}\right] f'''(0) + Rf. \tag{7.2}$$

This also occurred with the q.f. obtained from (4.2) of Theorem 5 for $m = 3$, which differs from (7.2) in that it uses the term $S'''(0)$ instead of $f'''(0)$. These q.f. correspond to quintic spline approximations, (7.2) using $I = \{1, 3\}$ and the latter using $I = \{1, 2\}$.

In regard to the absolute error, we consider

$$\int_0^\infty e^{-x} \cos xt \, dx = \frac{1}{1 + t^2}$$

and the step $h = 2\pi/32 \approx 0.2$ for the q.f. (7.1) and (7.2). We can compute bounds on Rf by using (22) and (23) and find that

$$| Rf |_3 \leqslant 4.6 \times 10^{-5}, \qquad | Rf |_5 \leqslant 1.8 \times 10^{-7} \quad \text{for } -16 \leqslant t \leqslant 16,$$

where the subscript indicates (7.1) or (7.2) respectively. Calculations using (7.1) indicate that the absolute error in the cubic case is actually greater than the bound for $| Rf |_5$ for values of t less than 2. For related observations and some graphs, see [13, pp. 98–106].

Acknowledgment

The results in this paper are contained in the author's doctoral thesis, which was written at the University of Wisconsin under the supervision of Professor I. J. Schoenberg. The author wishes to express his heartfelt appreciation to Professor Schoenberg for his valuable guidance and assistance.

References

1. B. Einarsson, Numerical calculation of Fourier integrals with cubic splines, *BIT* **8** (1968), 279–286.
2. B. Einarsson, On the calculation of Fourier integrals, preprint: IFIP Congress 71, Ljubljana — August 1971, Booklet TA-1, Numerical Mathematics, 99–103, North-Holland, Amsterdam.
3. M. Golomb, Approximation by periodic spline interpolants on uniform meshes, *J. Approximation Theory* **1** (1968), 26–65.
4. V. I. Krylov and N. S. Skoblya, "Handbook of Numerical Inversion of Laplace Transforms," (translation from Russian) Israel Program for Scientific Translations, Jerusalem, 1969.
5. M. Marsden and G. Taylor, Numerical evaluation of Fourier integrals, presented at Oberwolfach, June 1971.
6. I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, *Quart. Appl. Math.* **4** (1946), 45–99, 112–141.
7. I. J. Schoenberg, Cardinal interpolation and spline functions, *J. Approximation Theory* **2** (1969), 167–206.

8. I. J. SCHOENBERG, A second look at approximate quadrature formulae and spline interpolation, *Advances of Math.* **4** (1970), 277–300.
9. I. J. SCHOENBERG, Cardinal interpolation and spline functions II, Interpolation of data of power growth, *J. Approximation Theory* **6** (1972), 404–420.
10. I. J. SCHOENBERG, Cardinal interpolation and spline functions VI, Semi-cardinal interpolation and quadrature formulae, Mathematics Research Center Technical Summary Rep. No. 1180, Madison, Wisconsin, 1971. To appear in J. d'Analyse Math.
11. I. J. SCHOENBERG, Cardinal spline interpolation, CBMS Regional Conference Monograph, no. 12, SIAM, Philadelphia, 1973.
12. I. J. SCHOENBERG AND S. D. SILLIMAN, On semi-cardinal quadrature formulae, *Math. Comp.* **28** (1974), 483–497.
13. S. D. SILLIMAN, The numerical evaluation by splines of the Fourier transform and the Laplace transform, Mathematics Research Center Technical Summary Rep. No. 1183, Madison, Wisconsin, 1972.